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Free surface and surface tension effects on submerged bodies

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SUMMARY

The Oseen problem for the steady motion of an object beneath a free surface with surface tension under the action of gravity is formulated. The Green's tensor for the problem is used to convert the boundary value problem to a coupled pair of integral equations for the stresses which the fluid exerts on the object. For the special case of a flat plate, these integral equations are analyzed asymptotically for small velocities and deep immersion. This yields a Fredholm equation of the second kind with Cauchy kernel, which has a well-known solution. The results indicate the effect of surface tension on the stress singularities at the edges of the plate, and modification of the lift and drag due to the free surface.

1. Introduction

The object of this study is to investigate the effects of a free surface and surface tension on a moving body immersed in a viscous fluid. There is an extensive literature on the subject of bodies moving in a fluid of infinite extent, but due to the mathematical difficulties involved, very little work has been done with the free surface case. The results most closely related to the present work were obtained by Dugan [1]. He treated the problem of a point force moving under a free surface, which he then uses to obtain the equations pertinent to a submerged body, under the hypothesis of negligible surface tension. Other results concern the lift and drag on objects with and without fluid injection (extraction) in fluids of infinite extent. These may be found in a survey article by Olmstead and Gautesen [2], together with an extensive bibliography. Viscous and surface tension effects of a moving atmospheric disturbance were studied by Wu and Messick [3].

The approach of the paper is to transform the Oseen equation problem together with linearized boundary conditions to a pair of coupled integral equations for the stresses on an arbitrarily shaped two-dimensional body. This is achieved by first obtaining the Green's tensor for the problem. One of the advantages of this method is that the stresses on the body can be sought directly without having to solve for the velocity and pressure fields. When specialized to the case of a flat plate, the integral equations are solved asymptotically under the assumption of large immersion depth and low flow velocities. The results can then be compared with those obtained for plates immersed in fluids of infinite extent.

2. Formulation of the problem

We consider the steady two-dimensional flow of a viscous incompressible fluid past a body Σ

submerged a distance $\mathscr{H}$ below the free surface. The coordinate system is chosen such that the x-axis corresponds to the undisturbed free surface. The domain occupied by the fluid will be denoted by Ω. The governing equations of the flow are taken to be the Navier-Stokes equations

$$\nabla \cdot \mathbf{w} = 0,$$

$$2R\mathbf{w} \cdot \nabla \mathbf{w} = -\nabla p + \Delta \mathbf{w}, \quad (x,y) \in \Omega. \tag{2.1}$$

The velocity $\mathbf{w}(x,y) = u\mathbf{i} + v\mathbf{j}$ has been normalized by the reference value Q, taken to be the minimum phase velocity for simple harmonic surface waves in a non-viscous medium,

$$Q = (4gT/\rho)^{1/4}. \tag{2.2}$$

Here ρ is the fluid density, T the surface tension, and g the gravitational acceleration. The fluid viscosity will be denoted by μ. The Reynolds number R appearing in eq. (2.1) is given by

$$R = \frac{\rho \mathscr{L} Q}{2\mu}, \tag{2.3}$$

where the length scale $\mathscr{L}$ is a characteristic length of the body and the factor 2 appears for convenience. For the boundary conditions on solutions of eq. (2.1) we require that the velocity and pressure assume the values for the uniform flow at infinity,

$$\mathbf{w} \to \frac{U_0}{Q}\,\mathbf{i}, \quad p \to 0 \text{ as } x^2 + y^2 \to +\infty, \quad (x,y) \in \Omega, \tag{2.4}$$

while on the body

$$\mathbf{w} = \mathbf{0} \text{ on } \Sigma, \tag{2.5}$$

i.e. no fluid penetrates the body and no slip between body and fluid. On the free surface, described by

$$y = \eta(x), \tag{2.6}$$

the tangential stress is continuous, while the normal stress has a jump proportional to the surface tension and mean curvature. For a Newtonian fluid, this yields [4]

$$2\frac{d\eta}{dx}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \left[1 - \left(\frac{d\eta}{dx}\right)^2\right]\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

$$-\frac{1}{2}p + k\eta - \frac{1}{2}\frac{d\eta}{dx}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial y} = \tau \frac{d^2\eta}{dx^2}\left[1 + \left(\frac{d\eta}{dx}\right)^2\right]^{-3/2} \tag{2.7}$$

$$\text{on } y = \eta(x).$$

The constants $k = \rho g \mathscr{L}^2/(2\mu Q)$ and $\tau = T(2\mu Q)$ are a Reynolds number and the non-dimensionalized surface tension respectively. The requirement that no fluid particle cross the free surface gives rise to the kinematic condition

$$u \frac{d\eta}{dx} - v = 0 \text{ on } y = \eta(x). \tag{2.8}$$

For large immersion depth and low flow velocities, we expect the velocity and pressure fields to be well described by the Oseen equation together with the linearized boundary conditions known as the small amplitude approximation. More formally, by expanding the velocity and pressure fields in a small parameter, say $\epsilon = \mathscr{L}/\mathscr{H}$,

$$\begin{aligned} \mathbf{w} &= \frac{U_0}{Q}\,\mathbf{i} + \epsilon\bar{\mathbf{w}} + o(\epsilon), \\ p &= \epsilon\bar{p} + o(\epsilon), \\ \eta &= \epsilon\bar{\eta} + o(\epsilon), \end{aligned} \tag{2.9}$$

one finds that, to order ϵ, eq. (2.1), (2.4)-(2.8) may be replaced by

$$\nabla \cdot \mathbf{w} = 0, \quad y < 0, \tag{2.10}$$

$$2R \frac{\partial}{\partial x} \mathbf{w} = -\nabla p + \Delta \mathbf{w}, \quad y < 0, \tag{2.11}$$

$$\mathbf{w} \to u_0 R \mathbf{i}, \quad p \to 0 \text{ as } x^2 + y^2 \to +\infty, \quad y < 0, \tag{2.12}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \text{ on } y = 0, \tag{2.13}$$

$$-\frac{1}{2}\frac{\partial p}{\partial x} + \left(k + \frac{\partial^2}{\partial x \partial y} - \tau \frac{\partial^2}{\partial x^2}\right) v = 0 \text{ on } y = 0, \tag{2.14}$$

$$\frac{d\eta}{dx} = \frac{v}{u_0 R} \text{ on } y = 0, \tag{2.15}$$

$$\mathbf{w} = \mathbf{O} \text{ on } \Sigma. \tag{2.16}$$

The quantity u_0 in eq. (2.11) is given by $u_0 = U_0/(RQ)$ and is supposed to be of order $O(1)$ for small Reynolds number R. It may be pointed out that, as with the usual Oseen approximation, eqs. (2.9) cannot be a uniformly valid asymptotic approximation as long as the boundary condition $\mathbf{w} = \mathbf{O}$ is envoked on the body.

3. The Green's tensor

The Green's tensor E_{ij} for the system (2.10)-(2.16) may be obtained from the vector $\mathbf{W}(x,y|x_0,y_0) = W_1\mathbf{i} + W_2\mathbf{j}$ satisfying the equations

$$\nabla \cdot \mathbf{W} = 0, \quad y < 0,$$

$$2R\,\frac{\partial}{\partial x}\,\mathbf{W} = -\nabla P + \Delta\mathbf{W} + \mathbf{f}\delta(x,y|x_0,y_0), \quad y < 0,$$

$$\mathbf{W} \to \mathbf{0}, \quad P \to 0 \ \text{ as } x^2 + y^2 \to +\infty, \tag{3.1}$$

$$\frac{\partial W_1}{\partial y} + \frac{\partial W_2}{\partial x} = 0 \ \text{ on } y = 0,$$

$$-\frac{1}{2}\frac{\partial P}{\partial x} + \left(k + \frac{\partial^2}{\partial x \partial y} - \tau\frac{\partial^2}{\partial x^2}\right) W_2 = 0 \ \text{ on } y = 0.$$

This system of equations describes the flow field induced by a point force

$$\mathbf{f} = \Gamma\mathbf{i} + \Lambda\mathbf{j} \tag{3.2}$$

located at some point (x_0,y_0) within the fluid. For the special case $\mathbf{f} = \mathbf{i}$ (i.e. $\Gamma = 1, \Lambda = 0$), W_i reduces to the tensor component E_{1i}, $i \in \{1,2\}$. Similarly, the components E_{2i} are obtained by setting $\mathbf{f} = \mathbf{j}$. The differential equations in system (3.1) admit the well-known Oseen solutions $\mathbf{V} = \Gamma\mathbf{V}_1 + \Lambda\mathbf{V}_2$ and $P = \Gamma P_1 + \Lambda P_2$ given by

$$\mathbf{V}_1(x,y|x_0,y_0) = -\frac{1}{2\pi}\left\{\frac{x - x_0}{r^2} - R\exp(R(x - x_0))\left[\frac{x - x_0}{r}K_1(Rr) + K_0(Rr)\right]\right\}\mathbf{i}$$

$$-\frac{1}{2\pi}\left\{\frac{y - y_0}{r^2} - R\exp(R(x - x_0))\,\frac{y - y_0}{r}K_1(Rr)\right\}\mathbf{j},$$

$$\mathbf{V}_2(x,y|x_0,y_0) = -\frac{1}{2\pi}\left\{\frac{y - y_0}{r^2} - R\exp(R(x - x_0))\,\frac{y - y_0}{r}K_1(Rr)\right\}\mathbf{i}$$

$$+\frac{1}{2\pi}\left\{\frac{x - x_0}{r^2} - R\exp(R(x - x_0))\left[\frac{x - x_0}{r}K_1(Rr) - K_0(Rr)\right]\right\}\mathbf{j},$$

$$P_1(x,y|x_0,y_0) = \frac{R}{\pi}\,\frac{x - x_0}{r^2},$$

$$P_2(x,y|x_0,y_0) = \frac{R}{\pi}\,\frac{y - y_0}{r^2},$$

where $r^2 = (x - x_0)^2 + (y - y_0)^2$.

It was shown by Olmstead [5] and Dugan [1] that a solution $\mathbf{W}, P$ of the system (3.1) may be obtained by distributing the Oseen solutions on the free surface:

$$\mathbf{W}(x,y\,|x_0,y_0) = \Gamma\mathbf{V}_1(x,y\,|x_0,y_0) + \Lambda\mathbf{V}_2(x,y\,|x_0,y_0)$$

$$+\int_{-\infty}^{+\infty} dx'\mathbf{V}_1(x,y\,|x',y')\sigma_1(x') + \int_{-\infty}^{+\infty} dx'\mathbf{V}_2(x,y\,|x',y')\sigma_2(x'),$$

(3.4)

$$P(x,y\,|x_0,y_0) = \Gamma P_1(x,y\,|x_0,y_0) + \Lambda P_2(x,y\,|x_0,y_0)$$

$$+ \int_{-\infty}^{+\infty} dx'P_1(x,y\,|x',y')\sigma_1(x') + \int_{-\infty}^{+\infty} dx'P_2(x,y\,|x',y')\sigma_2(x'),$$

where the unknown functions $\sigma_1(x)$, $\sigma_2(x)$ have to be determined from the remaining boundary conditions in (3.1). Inspection of the integrals in (3.4) shows them to be convolutions. Substituting eq. (3.4) into (2.13) and (2.14) and Fourier transforming over the x variable, we obtain a coupled pair of algebraic equations

$$A_1(\omega)\hat{\sigma}_1(\omega) + A_2(\omega)\hat{\sigma}_2(\omega) = \Gamma A_3(\omega) + \Lambda A_4(\omega),$$

$$B_1(\omega)\hat{\sigma}_1(\omega) + B_2(\omega)\hat{\sigma}_2(\omega) = \Gamma B_3(\omega) + \Lambda B_4(\omega) \qquad (3.5)$$

for $\hat{\sigma}_1(\omega)$ and $\hat{\sigma}_2(\omega)$. The carets denote Fourier transforms and the coefficients are given by

$$A_1(\omega) = R,$$

$$A_2(\omega) = (\omega + i0)^{1/2}\left[(\omega - i0)^{1/2} - \frac{\omega - iR}{(\omega - 2iR)^{1/2}}\right],$$

$$A_3(\omega) = i\exp(i\omega x_0)\{-\omega\exp(y_0(\omega + i0)^{1/2}(\omega - i0)^{1/2})$$

$$+ (\omega - iR)\exp(y_0(\omega + i0)^{1/2}(\omega - 2iR)^{1/2})\},$$

$$A_4(\omega) = \exp(i\omega x_0)\,(\omega + i0)^{1/2}\{-(\omega - i0)^{1/2}\exp(y_0(\omega + i0)^{1/2}(\omega - i0)^{1/2})$$

$$+ \frac{\omega - iR}{(\omega - 2iR)^{1/2}}\exp(y_0(\omega + i0)^{1/2}(\omega - 2iR)^{1/2}\},$$

$$B_1(\omega) = \frac{i}{2}\,\omega(\omega + i0)^{1/2}\left\{(\omega - 2iR)^{1/2} - \frac{\omega - iR}{(\omega - i0)^{1/2}}\right\},$$

$$B_2(\omega) = \frac{i}{2}\,\omega\left\{-R + \frac{k + \tau\omega^2}{(\omega + i0)^{1/2}}\left[\frac{1}{(\omega - i0)^{1/2}} - \frac{1}{(\omega - 2iR)^{1/2}}\right]\right\}, \qquad (3.6)$$

$$B_3(\omega) = \frac{1}{2}\exp(i\omega x_0)\{[k + \tau\omega^2 + i(\omega - iR)(\omega + i0)^{1/2}(\omega - i0)^{1/2}]$$

$$\exp(y_0(\omega + i0)^{1/2}(\omega - i0)^{1/2})$$

$$-[k + \tau\omega^2 + i\omega(\omega + i0)^{1/2}]\exp(y_0(\omega + i0)^{1/2}(\omega - 2iR)^{1/2})\},$$

$$B_4(\omega) = \frac{1}{2}\exp(i\omega xp)\left\{\left[\omega(\omega - iR) - i(k + \tau\omega^2)\frac{(\omega + i0)^{1/2}}{(\omega - i0)^{1/2}}\right]\right.$$

$$\exp(y_0(\omega + i0)^{1/2}(\omega - i0)^{1/2})$$

$$\left. + \left[-\omega^2 + i(k + \tau\omega^2)\frac{(\omega + i0)^{1/2}}{(\omega - 2iR)^{1/2}}\right]\exp(y_0(\omega + i0)^{1/2}(\omega - 2iR)^{1/2})\right\},$$

where the notation $(\omega - i0)^{1/2}$, $(\omega + i0)^{1/2}$ is intended to keep track of the branch cuts which run along the positive and negative imaginary axis respectively. The cut for $(\omega - 2iR)^{1/2}$ is also taken along the imaginary axis. For $R \neq 0$, system (3.5) may be solved for $\hat{\sigma}_1(\omega)$ and $\hat{\sigma}_2(\omega)$, thereby obtaining the solution (3.4) of the Green's tensor problem in the form

$$\mathbf{W}(x,y\,|x_0,y_0) = \Gamma\mathbf{V}_1(x,y\,|x_0,y_0) + \Lambda\mathbf{V}_2(x,y\,|x_0,y_0)$$

$$+\frac{\Gamma}{\pi}\,Re\int_0^{+\infty} d\omega\mathbf{M}_1(\omega,x,y\,|x_0,y_0) + \frac{\Lambda}{\pi}\,Re\int_0^{+\infty} d\omega\mathbf{M}_2(\omega,x,y\,|x_0,y_0),$$

$$P(x,y\,|x_0,y_0) = \Gamma P_1(x,y\,|x_0,y_0) + \Lambda P_2(x,y\,|x_0,y_0) \tag{3.7}$$

$$+\frac{\Gamma}{\pi}\,Re\int_0^{+\infty} d\omega N_1(\omega,x,y\,|x_0,y_0) + \frac{\Lambda}{\pi}\,Re\int_0^{+\infty} d\omega N_2(\omega,x,y\,|x_0,y_0),$$

where the integrals describe the free surface effects. The integrands are given by

$$\mathbf{M}_\ell = M_{\ell 1}\mathbf{i} + M_{\ell 2}\mathbf{j},\ \ell \in \{1,2\}, \tag{3.8}$$

where

$$M_{11} = -\frac{i}{2}\exp(-i\omega(x - x_0))\left\{\exp(-|y + y_0|\omega)\left[1 - 2\frac{\omega(\omega - iR)^2}{D(\omega)}\right]\right.$$

$$-\exp(-|y + y_0|\omega^{1/2}(\omega - 2iR)^{1/2})\left[1 + 2\omega^2\frac{\omega^{1/2}(\omega - 2iR)^{1/2}}{D(\omega)}\right]\frac{(\omega - 2iR)^{1/2}}{\omega^{1/2}}$$

$$+\exp(-|y|\omega - |y_0|\omega^{1/2}(\omega - 2iR)^{1/2})\left[2\omega(\omega - iR)\frac{\omega^{1/2}(\omega - 2iR)^{1/2}}{D(\omega)}\right]$$

$$+\exp(-|y_0|\omega - |y|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega(\omega-iR)\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]\Bigg\}$$

$$M_{12} = -\frac{1}{2}\exp(-i\omega(x-x_0))\Bigg\{\exp(-|y+y_0|\omega)\left[1-2\,\frac{\omega(\omega-iR)^2}{D(\omega)}\right]$$

$$-\exp(-|y+y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[1+2\omega^2\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]$$

$$+\exp(-|y|\omega - |y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega^2\,\frac{(\omega-iR)}{D(\omega)}\right]$$

$$+\exp(-|y_0|\omega - |y|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega(\omega-iR)\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]\Bigg\}$$

$$M_{21} = -\frac{1}{2}\exp(-i\omega(x-x_0))\Bigg\{\exp(-|y+y_0|\omega)\left[1-2\,\frac{\omega(\omega-iR)^2}{D(\omega)}\right]$$

$$-\exp(-|y+y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[1+2\omega^2\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]$$

$$+\exp(-|y|\omega - |y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega(\omega-iR)\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]$$

$$+\exp(-|y_0|\omega - |y|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega^2\,\frac{(\omega-iR)}{D(\omega)}\right]\Bigg\}$$

$$M_{22} = -\frac{i}{2}\exp(-i\omega(x-x_0))\Bigg\{\exp(-|y+y_0|\omega)\left[1-2\,\frac{\omega(\omega-iR)^2}{D(\omega)}\right]$$

$$-\exp(-|y+y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[1+2\omega^2\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]\quad\frac{\omega^{1/2}}{(\omega-2iR)^{1/2}}$$

$$+\exp(-|y|\omega - |y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega^2\,\frac{(\omega-iR)}{D(\omega)}\right]$$

$$+\exp(-|y_0|\omega - |y|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega^2\,\frac{(\omega-iR)}{D(\omega)}\right]\Bigg\},$$

$$N_1 = iR\exp(-i\omega(x-x_0))\Bigg\{\exp(-|y+y_0|\omega)\left[1-2\,\frac{\omega(\omega-iR)^2}{D(\omega)}\right]$$

$$+\exp(-|y|\omega - |y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega(\omega-iR)\,\frac{\omega^{1/2}(\omega-2iR)^{1/2}}{D(\omega)}\right]\Bigg\},$$

$$N_2 = R\exp(-i\omega(x-x_0))\left\{\exp(-|y+y_0|\omega)\left[1-2\,\frac{\omega(\omega-iR)^2}{D(\omega)}\right]\right.$$

$$\left.+\exp(-|y|\omega-|y_0|\omega^{1/2}(\omega-2iR)^{1/2})\left[2\omega^2\,\frac{(\omega-iR)}{D(\omega)}\right]\right\}, \tag{3.9}$$

where the denominator

$$D(\omega) = R(k+\tau\omega^2)+\omega(\omega-iR)^2-\omega^2\omega^{1/2}(\omega-2iR)^{1/2}. \tag{3.10}$$

The shape of the surface wave is obtained from eq. (2.15):

$$\eta(x) = \frac{1}{u_0\pi}\,Re\int_0^{+\infty} d\omega\,\frac{\exp(-i\omega x_0)}{D(\omega)}\,\{\exp(-|y_0|\omega)\,[-(\omega-iR)\Gamma+i(\omega-iR)\Lambda]$$

$$+\exp(-|y_0|\omega^{1/2}(\omega-2iR)^{1/2})\,[\omega^{1/2}(\omega-2iR)^{1/2}\Gamma-i\omega\Lambda]\}. \tag{3.11}$$

The special case of zero surface tension was obtained by Dugan [1]. The limit as y_0 tends to 0^- describes a delta function stress on the free surface. The case of a delta function pressure disturbance ($\Gamma = 0$ and $\Lambda = -P_0$) was obtained by Wu and Messick [3] by a different method, which provides a check on the analysis so far.

4. The submerged plate

By distributing point forces $\mathbf{f}(x_0,y_0) = \Gamma(x_0,y_0)\mathbf{i} + \Lambda(x_0,y_0)\mathbf{j}$ along the body surface Σ, the Green's tensor may be used to express the velocity and pressure field for the submerged body problem (2.10)-(2.16) in terms of the unknown stresses on the body:

$$\mathbf{w}(x,y) = u_0Ri - \int_\Sigma ds_0\Gamma(x_0,y_0)\,\Big\{\mathbf{V}_1(x,y|x_0,y_0) - \frac{1}{\pi}\,Re\int_0^{+\infty} d\omega\mathbf{M}_1(\omega,x,y|x_0,y_0)\Big\}$$

$$-\int_\Sigma ds_0\Lambda(x_0,y_0)\,\Big\{\mathbf{V}_2(x,y|x_0,y_0) - \frac{1}{\pi}\,Re\int_0^{+\infty} d\omega\mathbf{M}_2(\omega,x,y|x_0,y_0)\Big\},$$

$$p(x,y) = -\int_\Sigma ds_0\Gamma(x_0,y_0)\,\Big\{P_1(x,y|x_0,y_0) - \frac{1}{\pi}\,Re\int_0^{+\infty} d\omega N_1(\omega,x,y|x_0,y_0)\Big\}$$

$$-\int_\Sigma ds_0\Lambda(x_0,y_0)\,\Big\{P_2(x,y|x_0,y_0) - \frac{1}{\pi}\,Re\int_0^{+\infty} d\omega N_2(\omega,x,y|x_0,y_0)\Big\}, \tag{4.1}$$

with the variables x_0, y_0 parameterized by the arclength s_0. The corresponding surface wave is given by

$$\eta(x) = \frac{1}{u_0\pi} \int_\Sigma ds_0 \Gamma(x_0,y_0) Re \int_0^{+\infty} d\omega \frac{\exp(i\omega x_0)}{D(\omega)} [(\omega - iR)\exp(-|y_0|\omega)$$

$$- \omega^{1/2}(\omega - 2iR)^{1/2}\exp(-|y_0|\omega^{1/2}(\omega - 2iR)^{1/2})]$$

$$+ \frac{1}{u_0\pi} \int_\Sigma ds_0 \Lambda(x_0,y_0) Re \int_0^{+\infty} d\omega \frac{\exp(i\omega x_0)}{D(\omega)} [-i(\omega - iR)\exp(-|y_0|\omega)$$

$$+ i\omega \exp(-|y_0|\omega^{1/2}(\omega - 2iR)^{1/2})].$$

Application of the remaining boundary conditions (2.16) on the body surface to equation (4.1) gives rise to a pair of coupled integral equations which may be solved for the stresses.

In particular, for a plate of length $2\mathscr{L}$ immersed at a depth $h = \mathscr{H}/\mathscr{L}$ in non-dimensional coordinates, and inclined at an angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ with respect to the x-axis, the resulting integral equations are

$$\begin{aligned} u_0 R = & \int_{-1}^{1} dt_0 \Gamma(t_0) [V_{11}(t_1,t_0) + \frac{R}{\pi} Re \int_0^{+\infty} d\Omega M_{11}(\Omega,t_1,t_0)] \\ & + \int_{-1}^{1} dt_0 \Lambda(t_0) [V_{12}(t_1,t_0) + \frac{R}{\pi} Re \int_0^{+\infty} d\Omega M_{12}(\Omega,t_1,t_0)], \\ 0 = & \int_{-1}^{1} dt_0 \Gamma(t_0) [V_{21}(t_1,t_0) + \frac{R}{\pi} Re \int_0^{+\infty} d\Omega M_{21}(\Omega,t_1,t_0)] \\ & + \int_{-1}^{1} dt_0 \Lambda(t_0) [V_{22}(t_1,t_0) + \frac{R}{\pi} Re \int_0^{+\infty} d\Omega M_{22}(\Omega,t_1,t_0)], \end{aligned} \tag{4.3}$$

where the variables of integration are

$$\begin{aligned} &\Omega = \frac{\omega}{R}. \\ &x_i = t_i \cos\theta, \\ &y_i = t_i \sin\theta - h, \quad i \in \{0,1\}, \end{aligned} \tag{4.4}$$

so that the variables t_i measure stress along the plate. The net stresses at some points t_0 are given by

$$\begin{aligned} &\Gamma(t_0) = \Gamma_+(x_0(t_0), y_0(t_0)) + \Gamma_-(x_0(t_0), y_0(t_0)), \\ &\Lambda(t_0) = \Lambda_+(x_0(t_0), y_0(t_0)) + \Lambda_-(x_0(t_0), y_0(t_0)), \end{aligned} \tag{4.5}$$

where the plus and minus sign indices refer to the upper and lower face of the plate respectively. The functions $V_{ij}(t_1,t_0), M_{ij}(t_1,t_0)$ are the components of the vectors $\mathbf{Y}_i$ and $\mathbf{M}_i$ of eq. (3.3) and (3.8) rewritten in the new variables. No explicit inverse for eq. (4.3) is known, but it is possible to do asymptotic expansions of the kernels for low Reynolds number flows and large immersion depth. By the method of steepest descents, only the critical point at the origin is found to contribute to the lowest order term for low values of the surface tension such that $\tau \ll 1$. The system of eq. (4.3) reduces to

$$\int_{-1}^{1} dt_0 \Gamma(t_0) \ln |t_1 - t_0| = 2\pi u_0 + \left(\ln \frac{2}{R} - \gamma + \cos^2\theta\right) D + \left(\sin\theta\cos\theta - \frac{1}{2Rh}\right) L + O(R \ln R) + o((Rh)^{-2}),$$

$$\int_{-1}^{1} dt_0 \Lambda(t_0) \ln |t_1 - t_0| = \left(\sin\theta\cos\theta + \frac{1}{2Rh}\right) D + \left(\ln \frac{2}{R} - \gamma - \cos^2\theta\right) L + O(R \ln R) + o((Rh)^{-2}), \tag{4.6}$$

where D and L, the drag and lift exerted on the plate, are defined by

$$D = \int_{-1}^{1} dt_0 \Gamma(t_0), \quad L = \int_{-1}^{1} dt_0 \Lambda(t_0) \tag{4.7}$$

respectively, and $\gamma = .577\ldots$ is Euler's constant. Equations (4.6) are known as Carleman equations. Their solution is given by

$$\Gamma(t) = \frac{D}{\pi} (1 - t^2)^{-1/2}, \quad \Lambda(t) = \frac{L}{\pi} (1 - t^2)^{-1/2}, \tag{4.8}$$

with

$$D = 2\pi u_0 \frac{\ln \frac{4}{R} - \gamma - \cos^2\theta}{\left(\ln \frac{4}{R} - \gamma\right)^2 - \cos^2\theta + \left(\frac{1}{2Rh}\right)^2},$$

$$L = 2\pi u_0 \frac{-\sin\theta\cos\theta - \frac{1}{2Rh}}{\left(\ln \frac{4}{R} - \gamma\right)^2 - \cos^2\theta + \left(\frac{1}{2Rh}\right)^2}. \tag{4.9}$$

In the limit as the immersion depth goes to infinity, these expressions reduce to

$$D_\infty = 2\pi u_0 \frac{\ell n \dfrac{4}{R} - \gamma - \cos^2\theta}{\left(\ell n \dfrac{4}{R} - \gamma\right)^2 - \cos^2\theta},$$

$$L_\infty = 2\pi u_0 \frac{-\sin\theta\cos\theta}{\left(\ell n \dfrac{4}{R} - \gamma\right)^2 - \cos^2\theta}, \tag{4.10}$$

which are identical to those given in Olmstead and Gautesen [1] for the forces experienced by a flat plate moving in a fluid of infinite entent.

For surface tension in the range $1 \ll \tau \ll kR^{-2}$, with $\tau^3 \ll kh^2$, additional terms appear in the asymptotic expansions of eq. (4.3). These now become

$$\int_{-1}^{1} dt_0 \Gamma(t_0) \left[\ell n|t_1 - t_0| - 2\pi \frac{k}{R^2\sigma^2} H(t_1 - t_0)\right] = -2\pi u_0 + \left(\ell n \frac{2}{R} - \gamma + \cos^2\theta\right) D$$

$$+ \left(\sin\theta\cos\theta - \frac{1}{2Rh}\right) L + O(R\,\ell n\,R) + o((Rh)^{-2}, k(R\sigma)^{-2}),$$

$$\int_{-1}^{1} dt_0 \Lambda(t_0) \left[\ell n|t_1 - t_0| - 2\pi \frac{k}{R^2\sigma^2} H(t_1 - t_0)\right] = \left(\sin\theta\cos\theta + \frac{1}{2Rh}\right) D \tag{4.11}$$

$$+ \left(\ell n \frac{2}{R} - \gamma - \cos^2\theta\right) L + O(R\,\ell n\,R) + o((Rh)^{-2}, k(R\sigma)^{-2}),$$

with solution

$$\Gamma(t) = D\left(\pi^2 + \frac{4\pi^2 k^2}{(R\tau)^4}\right)^{-1/2} (1-t)^{-p}(1+t)^{p-1},$$

$$\Lambda(t) = L\left(\pi^2 + \frac{4\pi^2 k^2}{(R\tau)^4}\right)^{-1/2} (1-t)^{-p}(1+t)^{p-1}, \tag{4.12}$$

$$p = \frac{1}{\pi}\arctan\frac{R^2\tau^2}{2k}.$$

The lift and drag are given by

$$D = 2\pi u_0 \frac{\ell n \dfrac{2}{R} + c - \gamma - \cos^2\theta}{\left(\ell n \dfrac{2}{R} + c - \gamma\right)^2 - \cos^2\theta + \left(\dfrac{1}{2Rh}\right)^2}, \tag{4.13}$$

$$L = 2\pi u_0 \frac{-\sin\theta\cos\theta - \dfrac{1}{2Rh}}{\left(\ell n \dfrac{2}{R} + c - \gamma\right)^2 - \cos^2\theta + \left(\dfrac{1}{2Rh}\right)^2},$$

$$c = -\gamma - \ell n\, 2 - \psi(p),$$

where ψ is the digamma function.

5. Conclusion

Comparing eqs. (4.9) and (4.10), we see that the drag in a semi-infinite fluid in the low Reynolds number regime is less than the drag for an infinite fluid. This would indicate that the energy stored in the surface wave is less than the energy stored in the additional fluid above the curve corresponding to the free surface when the fluid is infinite. This is the opposite of the results for the high Reynolds number regime, where the drag in the semi-infinite fluid is known to be larger than the drag in the infinite fluid.

We also see that the lift is enhanced by the presence of the free surface. This can be explained as a tendency of the fluid to pass underneath the body rather than between the body and the free surface.

In the large surface tension case, the character of the singularities at the edges of the plate are altered, the singularity at the leading edge becoming more pronounced compared to the low surface tension, while the stress at the trailing edge becomes correspondingly less singular.

The Green's function method used is readily extended to the case of a body capable of injecting or extracting fluid from its surroundings. The results are similar to the expressions obtained above and reduce to the infinite fluid results given in [2] in the limit of infinite immersion depth.

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